A SURVEY AND SELECTED RESULTS ON INVERSE DOMINATION IN GRAPHS

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ABSTRACT

Historically, the concept of domination in graphs found its origin in 1850s with the interest of several chess players. Among the numerous applications of the domination theory in graphs, the most often discussed is a communication network. This network consists of communication links between a fixed set of sites. The problem is to select a smallest set of sites at which the transmitters are placed so that every other site in the network is joined by a direct communication link to the site, which has a transmitter. In other words the problem is to find a minimum dominating set in the graph corresponding to this network. Kulli and Sigarkanti [13] considered the problem of selecting two disjoint sets of transmitting stations so that one set can provide service in the case of failure of some of the transmitting stations of the other set. This led them to define the inverse domination number \( \gamma^{-1}(G) \). Let \( D \) be a \( \gamma \)-set of \( G \). If \( D' \subseteq V(G) - D \) is another \( \gamma \)-set of \( G \) then \( D' \) is called an inverse dominating set with respect to \( D \). The minimum cardinality of the set \( D \) is called an inverse domination number of \( G \) and is denoted by \( \gamma^{-1}(G) \). In any network (or graphs), dominating sets are central sets and play a vital role in routing problems in parallel computing. Suppose \( D \) is a \( \gamma \)-set in a graph (or network) \( G \), when the network fails in some nodes in \( D \), the inverse dominating set \( D' \) will take care of the role of \( D \). In this aspect, it is worthwhile to concentrate on dominating and inverse dominating sets. In this paper we offer a survey of selected recent results on inverse domination in graphs.

KEYWORDS: Domination, Inverse domination.

SUBJECT CLASSIFICATION NUMBER: AMS - 05C69, 05C70

INTRODUCTION

Recently several authors initiated the study of inverse domination in graphs. In this paper we restricted our focus to recent selected theoretical results of inverse domination for some classes of graphs.

1.1 BASIC TERMINOLOGY AND CONCEPTS OF GRAPH THEORY:

We consider connected, undirected, finite graphs without loops. We follow the notations and terminology of Harary [8]. Let \( G = (V, E) \) be a graph with \( |V| = p \) and \( |E| = q \). \( n \) denotes number of blocks of \( G \). The maximum degree of a vertex in \( G \) is \( \Delta(G) \). The minimum degree among the vertices of \( G \) is denoted by \( \min deg(G) \) or \( \beta(G) \). The vertex independence number \( \beta_q(G) \) of a graph \( G \) is the maximum cardinality of an independence set of vertices in \( G \). The greatest distance between any two vertices of a connected graph \( G \) is called the diameter of \( G \) and is denoted by \( \text{diam}(G) \). Let \( G = (V, E) \) be a graph. A set \( D \subseteq V \) is said to be a dominating set of \( G \) if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The minimum cardinality of the set \( D \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). For more details see [2], [14], [18],[22]. Let \( D \) be a minimum dominating set in a graph \( G \). If \( V \) - \( D \) contains a dominating set \( D' \) of \( G \), then \( D' \) is called an inverse dominating set with respect to \( D \). An inverse dominating set \( D \) is called a minimum inverse dominating set, if \( D \) consists of minimum number of vertices among all inverse dominating sets. The number of vertices in a minimum inverse dominating set is defined as the inverse domination number of a graph \( G \), and it is denoted by \( \gamma^{-1}(G) \). This concept was introduced by Kulli V.R. and Sigarkanti S.C. [13]. Also G. S. Domke et al. [6], [7] characterized the graphs which satisfy \( \gamma(G) + \gamma^{-1}(G) = n \). In their work, they gave a lower bound for the inverse domination number for trees. They also provided a constructive characterization for the trees which
achieve their lower bound. T Tamizh Chelvam, T Asir and G. S. Grace Prema [19], [20], [21] consider graphs $G$ with $n$ vertices for which $\gamma(G) = \gamma^{-1}(G)$. They derived some bounds for $\gamma^i(G)$ and $\gamma(G)$ through $n$ and characterize the graphs with $\gamma(G) = \gamma^{-1}(G) = n$.

Also they characterize the graphs with $\gamma(G) = \gamma^{-1}(G) = n-1$ and characterized graphs with minimum degree at least two for which the sum of their domination number and inverse domination number is $n - 1$. Further, they constructed classes of graphs with minimum degree one for which the sum of their domination number and inverse domination number is $n-1$ and finally they characterize all graphs for which the sum of domination number and inverse domination number is $n-1$.

**BASIC RESULTS ON INVERSE DOMINATION**

Proposition 1[13]: If a graph $G$ has no isolated vertices, then $\gamma(G) \leq \gamma^{-1}(G)$.

Proposition 2[13]: If a graph $G$ has no isolated vertices, then $\gamma(G) + \gamma^{-1}(G) \leq p$.

Theorem 1[13]: Let $D$ be a minimum dominating set of $G$. If for every vertex $v \in D$, the induced subgraph $\langle N[v]\rangle$ is a complete graph of order at least 2, then $\gamma(G) = \gamma^{-1}(G)$.

Theorem 2[13]: Let $\tau$ denote the family of minimum dominating sets of $G$. If every minimum dominating set $D \subseteq \tau$. $V - D$ is independent, then $\gamma(G) + \gamma^{-1}(G) = p$.

Theorem 3[13]: If every nonend vertex of a tree $T$ is adjacent to at least one end vertex, then, $\gamma(G) + \gamma^{-1}(G) = p$.

Theorem 4[13]: Let $G$ be a connected graph with $\delta(G) \geq 1$, then $\gamma(G) + \gamma^{-1}(G) = p$ if and only if $G = C_p$.

Theorem 5[13]: Let $G$ be a connected graph with $p \geq 3$ and $\delta(G) = 1$. Let $L \subseteq V$ be the set of all leaves and let $S = N(L)$ (stems). Then $\gamma(G) + \gamma^{-1}(G) = p$ if and only if the following two conditions hold.

(i) $V - S$ is an independent set and (ii) For every set $x \in V - (S \cup L)$, every stem in $N(x)$ is adjacent to at least two leaves.


Theorem 6[13]: For any graph $G$ with no isolated vertices, $\gamma(G) + \gamma^{-1}(G) = p$ if and only if each component of $G$ is either $C_2$, $K_2$ or a graph described in Theorem 4.

Theorem 7[13]: If $G$ is a $(p, q)$ graph with $\gamma(G) = \gamma^{-1}(G)$, then $\frac{2p - q}{3} \leq \gamma^{-1}(G)$.

Theorem 8[13]: If a $(p, q)$ graph $G$ has no isolated vertices, then $\frac{2p - q}{3} \leq \gamma^{-1}(G)$.

Corollary 1[13]: For any tree $T$ of order $p \geq 2$, $\frac{p+1}{3} \leq \gamma^{-1}(T)$.

Kulli and Sigarkanti conjecture[11,13]:

For any graph $G$ without isolated vertices, $\gamma^{-1}(G) \leq \beta'_0(G)$.

Theorem 9[13]: Let $G$ be any graph without isolated vertices. If $\beta'_0(G) = \Gamma(G)$, then $\gamma^{-1}(G) \leq \beta'_0(G)$.

Conjecture 2[6]: If $G$ is a graph with no isolated vertex, then $\gamma^{-1}(G) \leq \alpha(G)$. Tamizh Chelvam and Grace Prema [21] characterized graphs for which $\gamma(G) = \gamma^{-1}(G) = (n - 1)/2$. In this context, they attempted to characterize graphs $G$ with $\delta(G) \geq 2$ for which $\gamma(G) + \gamma^{-1}(G) = n - 1$. They proved the following theorem.

Theorem 10[21]: Let $G$ be a connected graph on $n$ vertices with $\delta(G) \geq 2$. Then $\gamma(G) = \gamma^{-1}(G) = n - 1$ implies that $\gamma(G) = \gamma^{-1}(G)$.

**INVERSE DOMINATION ON SOME SPECIAL CLASSES OF GRAPHS**

3.1 CLAW-FREE GRAPHS:

Theorem 11[1]: If $G$ is a graph with $\delta(G) \geq 1$ satisfying $\gamma(G) = \iota(G)$, then $\gamma^{-1}(G) \leq \alpha(G)$.

Since every claw-free graph $G$ satisfies $\gamma(G) = \iota(G)$ we get the following corollary.

Corollary 2[1]: If $G$ is a claw-free graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \alpha(G)$.

Observation 1[1]: If a graph $G$ without isolated vertices satisfies $\gamma(G) = \alpha(G)$, then $G$ has two vertex-disjoint $\gamma(G)$-sets.
3.2. BIPARTITE AND CHORDAL GRAPHS

Observation 2[1]: If $G$ is a graph with $\tilde{\gamma}(G) \geq 1$, then $\gamma^{-1}(G) \leq \Gamma(G)$ and $a(G) \leq \Gamma(G)$.

Observation 3[1]: If $G$ is a graph with $\tilde{\gamma}(G) \geq 1$ and $a(G) = \Gamma(G)$, then $\gamma^{-1}(G) \leq a(G)$.

Corollary 3[1]: If $G$ is a bipartite graph, a chordal graph, a circular arc graph, a permutation graph, or a comparability graph with $\tilde{\gamma}(G) \geq 1$, then $\gamma^{-1}(G) \leq a(G)$.

3.3. VERY WELL-COVERED GRAPHS

A graph $G$ of order $n$ is said to be very well-covered if $\mu(G) = a(G) = n/2$. It is shown that Conjecture 1 is true for every very well-covered graph. It is also proved that every graph that has a perfect matching and has independence number one-half its order satisfies Conjecture 1.

Theorem 12[1]: If $G$ is a graph of order $n$ that has a perfect matching and $a(G) = n/2$, then $\Gamma(G) = n/2$.

Corollary 4[1]: If $G$ is a graph of order $n$ that has a perfect matching and $a(G) = n/2$, then $\gamma^{-1}(G) \leq a(G)$.

Since every very well-covered graph has a perfect matching the following consequence of Corollary 4.

Corollary 5[1]: If $G$ is a very well-covered graph of order $n$, then $\gamma^{-1}(G) \leq a(G)$.

3.4. NEARLY BIPARTITE GRAPHS AND SPLIT GRAPHS

A graph $G$ is called a split graph if its vertices can be partitioned into two sets $X$ and $Y$ such that $X$ is an independent set and $G[Y]$ is a complete graph.

Theorem 13[1]: If $G$ is a graph without isolated vertices that can be obtained from a bipartite graph by adding edges to one of its partite set $Y$ such that each component of $G[Y]$ is a complete graph, then $\gamma^{-1}(G) \leq a(G)$.

Since Conjecture 2 is true for split graphs the corollary 6 follows.

Corollary 6[1]: If $G$ is a split graph with no isolated vertex, then $\gamma^{-1}(G) \leq a(G)$.

Allan Frendrup, Michael A. Henning, Bert Randerath and Preben Dahl Vestergaard gave an algorithm for generalized cactus graph $G$, that can be used to construct an inverse dominating set with at most $a(G)$ vertices. In the algorithm they constructed an independent set $I$ and a set $S'$ such that $(S' \cup I)$ is a minimum inverse dominating set of a digraph $D$.

Theorem 14[1]: If $G \cong K_{1}$ is a generalized cactus graph, then $\gamma^{-1}(G) \leq a(G)$.

3.5. DIGRAPH

Definition 1[8]: A directed graph (digraph) $D$ is a pair $\langle V, A \rangle$, where $V$ is a finite nonempty set and $A$ is a subset of $V \times V - \{(x, x) / x \in V \}$. The elements of $V$ and $A$ are respectively called vertices and arcs. If $a$ is an arc, and $u$ and $v$ are vertices such that $(u, v) \in A$, then $a$ is said to join $u$ and $v$. $a$ is the tail (initial vertex) of $A$ and $v$ is the head (terminal vertex) of $A$.

INVERSE DOMINATION IN DIGRAPHS:

Let $D = (V, A)$ be a digraph. Let $S$ be a minimum dominating set in a digraph $D$. If $V - S$ contains a dominating set $S'$ of $D$, then $S'$ is called an inverse dominating set with respect to $S$. The minimum cardinality of an inverse dominating set of a digraph $D$ is called the inverse domination number of $D$ and is denoted by $\gamma^{-1}(D)$.

Definition 2[12]: The upper inverse domination number $\Gamma^{-1}(D)$ of a digraph $D$ is the maximum cardinality of an inverse dominating set of $D$.

Proposition 3[12]: For any directed cycle $C_{2p}$, $p \geq 2$,

$\gamma^{-1}(C_{2p}) = p$.

A $\gamma^{-1}$-set is a minimum inverse dominating set of a digraph $D$.

Proposition 4[12]: If a digraph $D$ has a $\gamma^{-1}$-set, then $\gamma(D) \leq \gamma^{-1}(D)$ ———— (1)

and this bound is sharp.

Proof: Clearly every inverse dominating set of a digraph is a dominating set. Thus (1) holds.
The directed cycles $C_{2p}$, $p \geq 2$ achieve this bound.

Proposition 5[12]: If a digraph $D$ has a $y'$-set, then $\gamma(D) + y'(D) \leq p$  (2) and this bound is sharp.

Proof: (2) follows from the definition of $y'(D)$.

The directed cycles $C_{2p}$, $p \geq 2$ achieve this bound.

3.6. SEMITOTAL BLOCK GRAPH

Definition 3[17]: Let $B = \{B_1, B_2, ..., B_n\}$ be the set of blocks of $G$. The semi-total block graph $T_b(G)$ of a graph $G$ is the graph whose point set is $V(G) \cup B(G)$ in which any two points are either adjacent or the corresponding members of $G$ are incident. The points and blocks of $G$ are members of $T_b(G)$.

A non-empty set $D \subseteq V \cup B$ is a dominating set of $T_b(G)$ if every point in $(V \cup B)-D$ is adjacent to at least one point in $D$ (Muddebihal, M.H. et al 2004). The domination number of $T_b(G)$ is denoted by $\gamma(T_b(G))$ and it is defined as the minimum cardinality taken over all the minimal dominating sets of $T_b(G)$.

Ameenal Bibi, K., Selvakumaran R. in 2010 studied inverse domination in Semitotal block graphs [3].

Let $D$ be the minimum dominating set of $T_b(G)$. If $(V \cup B)-D$ contains a dominating set $D'$ then $D'$ is called the Inverse dominating set of $T_b(G)$. The Inverse domination number in semi-total block graph is denoted by $\gamma'[T_b(G)]$ and it is defined as the minimum cardinality taken over all the minimal Inverse dominating sets of $T_b(G)$.

Observation 2[3]: For any path $P_n$ with $n \geq 2$ points, $\gamma'[T_b(P_n)] = \frac{n}{2}$.

Observation 3[3]: For any non-separable graph $G$, $\gamma'[T_b(G)] = 1$.

Theorem 15[3]: For any tree $T$, $\gamma'[T_b(T)] \geq \gamma'(T)$.

Theorem 16[3]: For any graph $G$ with $p$ blocks $y'[T_b(G)] \leq p$.

Theorem 17[3]: If $\gamma'(G) \leq \gamma'[T_b(G)]$ then $\gamma'(G) \leq p$, where $p$ is the number of blocks of $G$.

Theorem 18[3]: For any connected graph $G$,

$\gamma'(T_b(G)) \geq \frac{n}{2}$ where $n$ is the number of vertices of $G$.

Theorem 19[3]: For any graph $G$, $\gamma'[T_b(G)] \leq n - \delta(G)$.

Theorem 20[3]: For any graph $G$, $\gamma'[T_b(G)] \leq \beta_0(G)$ where $\beta_0(G)$ is the independence number of $G$.

Theorem 21[3]: For any graph $G$ without end points, $\gamma'[T_b(G)] \leq \text{diam}(G)$.

Theorem 22[3]: For any connected graph $G$ with $n \geq 4$ points,

(i) $\gamma'[T_b(G)] + y'[T_b(G)] \leq 2n - 1$.

(ii) $\gamma'[T_b(G)] + y'[\overline{T_b(G)}] \leq n + 1$.

Theorem 23[3]: For any connected graph $G$ with $p \geq 2$ blocks,

(i) $\gamma'[T_b(G)] + y'[\overline{T_b(G)}] \leq 2p$.

(ii) $\gamma'[T_b(G)] + y'[\overline{T_b(G)}] \leq p^2$.

3.7. JUMP GRAPH

Inverse domination on Jump Graphs was studied in [11] by Karthikeyan M., Elumalai A. They obtained exact values for some standard graphs and investigated interesting relations between inverse domination of Jump graphs and other parameters.

Definition 4[5]: The line graph $L(G)$ of $G$ has the edges of $G$ as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in $G$. They call the complement of line graph $L(G)$ as the jump graph $J(G)$ of $G$, found in [5]. The jump graph $J(G)$ of a graph $G$ is the graph defined on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in $G$. Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph $G$.

Remark 1: The isolated vertices of $G$ (if $G$ has) play no role in line graph and jump graph transformation. Here they assume that the graph $G$ under consideration is non-empty and has no isolated vertices found in [5].

Definition 5: Let $J(G) = (V, E)$ be a jump graph. Let $D$ be a minimum dominating set in a jump Graph $J(G)$. If $V - D$ contains a dominating set $D'$ of $J(G)$, then $D'$ is called an inverse dominating set with respect to $D$. The minimum cardinality of an inverse dominating set of a jump graph $J(G)$ is called the inverse domination number of $J(G)$ and it is denoted by $\gamma'[J(G)]$.

For any graph $G$, with $p \leq 4$, the jump graph $J(G)$ of $G$, is disconnected. Since only the connected jump graph is studied, $p > 4$ [10].

Theorem 24[11]:

1. For any path $P_p$ with $p \geq 5$, $\gamma'[J(P_p)] = 2$

2. For any Cycle $C_p$, with $p \geq 5$, $\gamma'[J(C_p)] = 2$
Theorem 45[4]: For any cubic bipartite graph, \( \gamma^{-1}(G) \leq \frac{D}{3} \).

3.8. SQUARE GRAPHS

Definition 6[9]: The square of a graph \( G \) denoted by \( G^2 \), has the same vertices as in \( G \) and the two vertices \( u \) and \( v \) are joined in \( G^2 \), if and only if they are joined in \( G \) by a path of length one or two. The concept of squares of graphs was introduced in [9].

M. H. Muddhebhal and Srinivasa G [15] established the following results on inverse domination for square graphs

Let \( D \) be a minimum dominating set of a square graph \( G^2 \). If \( V(G^2) \) contains another dominating set \( D' \) of \( G^2 \), then \( D' \) is called an inverse dominating set with respect to \( D \). The minimum cardinality of vertices in such a set is called an inverse domination number of \( G^2 \) and is denoted by \( \gamma^{-1}(G^2) \)

Theorem 25[11]: For any connected graph \( G \), \( \gamma^{-1}(J(G)) \geq 2 \).

Theorem 26[11]: For any connected \( (p, q) \) graph, \( \gamma^{-1}(J(G)) \leq q - \beta(G) + 1 \).

Theorem 27[11]: For any connected graph \( G \) with diameter, \( \text{diam}(G) \geq 2 \), \( \gamma^{-1}(J(G)) \geq 2 \).

Theorem 28[11]: For any connected graph \( G \) with circumference, \( \text{circ}(G) \geq 4 \), \( \gamma^{-1}(J(G)) \geq 2 \).

Theorem 29[11]: For any tree \( T \) with diameter greater than 3, \( \gamma^{-1}(J(G)) \geq 2 \).

Theorem 30[11]: For any connected \((p, q)\) graph \( G \), \( \gamma^{-1}(J(G)) < q - \Delta(G) \) where \( \Delta(G) \) is the maximum degree of \( G \).

Theorem 31[11]: For any connected graph \( G \) without pendant vertex \( \gamma^{-1}(J(G)) \leq \delta(G) \).

3.9. CUBIC BIPARTITE GRAPHS

Definition 7[8]: A graph is called a cubic graph (regular graph of degree 3) if the degree of each vertex is of degree three. A bipartite graph is a graph in which the vertex set can be divided into two disjoint vertex subsets \( X \) and \( Y \) such that every edge has one end in \( X \) and another end in \( Y \). A graph \( G \) is said to be cubic bipartite if it is bipartite as well as cubic.

V. Anandkumar, Radha Rajamani Iyer [4] discussed the relation between inverse domination number and independence number of a finite simple connected graphs.

Theorem 41[4]: An independent set of a graph \( G \) is an inverse dominating set if and only if it is maximal in \( (V - D) \).

Theorem 42[4]: If \( G \) is a complete graph of order \( p \geq 2 \), then \( \gamma^{-1}(G) = \beta(G) \).

Theorem 43[4]: If every non-end vertex of a graph \( G \) is adjacent to at least one end vertex, then \( \gamma^{-1}(G) = \beta(G) \).

Theorem 44[4]: In any simple graph \( G \), the addition of an edge decreases the number of minimal inverse dominating sets.

Theorem 45[4]: For any cubic bipartite graph, \( \gamma^{-1}(G) \leq \frac{D}{3} \).
INVERSE BLOCK DOMINATION IN GRAPHS

Numerous applications of inverse domination and recent research on different classes of graphs created an interest which has triggered us to define inverse block domination and initiate work on inverse domination in block graphs. A maximal induced sub graph without cut vertex is called a block of G. For any (p, q) graph G, a block graph is the graph whose vertices correspond to the blocks of G and two vertices in B(G) are adjacent whenever the corresponding blocks contain a common cut vertex in G. Block domination is introduced by M. H. Muddebihal et.al [16]. A set D ⊆ V(B(G)) is said to be a dominating set of B(G) if every vertex in V(B(G)) – D is adjacent to some vertex in D. The minimum cardinality of the set D is called the block domination number of G and is denoted by γ(B(G)).

If (V(B(G)) – D) contains another dominating set say D’, then D’ is called the inverse block dominating set of G with respect to D. The minimum cardinality of the set D’ is called inverse block domination number and is denoted by γ−1[B(G)]. We obtain inverse block domination number of some standard graphs, which are straight forward in Theorem 4.1.

Theorem 4.1: 1. For any path Pp with p ≥ 3, γ−1[B(Pp)] = \left[\frac{p}{3}\right]
2. For any star Kl,p with p ≥ 3, γ−1[B(Kl,p)] = 1

Proposition 4.1: If G has exactly one cutvertex incident with n ≥ 2 number of blocks, then γ−1[B(G)] = 1.

CONCLUSION

In this paper we surveyed some basic results of inverse domination in graph G and some special classes of graphs that involve components of G. Block domination and inverse block dominations may have applications in war planning as each vertex of B(G) represents a group of vertices of G and dominating groups are in contact with all other groups. In this regard we obtain inverse block domination number for some standard graphs and some more interesting results are in progress.

REFERENCES


